Exercise 4

Refer to Example 3 and solve (1) using $a = 1$, $b = 1$, $c = 1$, $d = -1$.

Solution

Equation (1) is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{1}$$

Make the change of variables, $\alpha = ax + bt = x + t$ and $\beta = cx + dt = x - t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(-1) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \\ &= \left(\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right) + \left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\ &= 2\frac{\partial u}{\partial \alpha}. \end{aligned}$$

Divide both sides by 2.

$$\frac{\partial u}{\partial \alpha} = 0$$

Integrate both sides partially with respect to α to get u.

$$u(\alpha, \beta) = f(\beta)$$

Here f is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x, t) = f(x - t)$$